ADDRESSING ALTERNATIVE CONCEPTIONS IN MATHEMATICS USING DISCREPANT EVENTS

TIERNEY KENNEDY
Kennedy Press
tierney@kennedypress.com.au

Abstract. The following report presents an analysis of eight mathematics lessons in which students displaying alternative conceptions were lead to reconceptualise their own understanding. Similarities were noted between the approach used by the teacher for juxtaposing discrepant events with questioning in each of these incidents and Cognitive Change Theory, an approach used by science educators to address scientific misconceptions by creating cognitive conflict. Initial data indicates that this approach may be applicable during the explore phase of challenging lessons, encouraging student to think through their own ideas and accommodate new information.

Introduction

Educators often consider why students answer questions in mathematics incorrectly, particularly when those answers are commonly given by students in a wide variety of circumstances (Swan, 2001). While some incorrect answers are simply errors or miscalculations, others are thought to be set within deeper levels of knowledge and more problematic for learners to overcome (Ryan & Williams, 2007). The term *misconception* is used by some researchers to describe situations in which a learner's understanding is considered to be in conflict with accepted meanings and understandings of mathematics (Barmby, Bilesborough, Harries, and Higgins, 2009). Other researchers object to this term because, while these ideas are technically incorrect, from the student's viewpoint, the ideas expressed are logical (Sneider and Ohadi, 1998). In this paper the term *alternative conceptions* is used, expressing the viewpoint that these ideas form a natural part of the development of mathematical understanding (Swan, 2001). This paper reflects the viewpoint held by Hansen (2014), that rather than trying to avoid the development of these ideas by students in the first place, effective teaching identifies, exposes and confronts these ideas, enabling students to restructure their own thinking.

One important assumption underpinning this paper is that students actively construct their own understanding of mathematics and that, to be effective, students need to think deeply about mathematics, connect ideas and be challenged. This paper espouses a connectionist (Askew, Brown, Rhodes, Johnson & William, 1997) or parapositional (Adam and Chigeza, 2014) disposition, that effective teachers focus heavily on the connections between concepts and use multiple teaching approaches as appropriate to the context.

A second important assumption is that when new information fits with what we already know it is assimilated within our existing conceptual understanding, but when it does not, we either reject the new information or accommodate it by changing our cognitive structures (Piaget & Inhelder, 1969). Posner, Strike, Hewson & Gertzog (1982) posited a theory for conceptual change in which an alternative conception which conflicts with new information, is modified or is no longer considered useful and is rejected as untenable.

This paper seeks to apply the lens of conceptual change theory to several mathematics lessons in which students displaying alternative conceptions were led to reconceptualise their ideas.

Background

Hattie (2009) expresses the belief that a learner's construction of knowledge and ideas is more important than the knowledge or ideas themselves. This construction, termed conceptual understanding by Bereiter (2002), connects both surface and deep knowledge to create a schema by which a learner interprets new ideas. Wenning (2008) explains that learners interpret new experiences and information in the light these existing schema, grafting new understandings onto prior conceptions. He theorizes that when new concepts do not fit within a learner's schema these are likely to be forgotten or even rejected, leading to his conclusion that addressing a learner's alternative conceptions in science is critical for the development of understanding.

Conceptual Change Theory (Posner et al., 1982), proposes a process by which a learner's existing conceptions may be replaced by more robust ideas. According to this theory, new information which conflicts with a learner's pre-existing schema is introduced to create cognitive conflict or disequilibrium (Piaget & Inhelder, 1969; Resnick, 1983), so that "the learner recognizes inconsistencies between existing beliefs and observed events" (Swan, 2005, p.7). As these inconsistencies are recognized, learners choose which of their ideas make the most sense and accommodate those that conflict by constructing new connections and changing their conceptual understanding (Piaget & Inhelder, 1969).

One approach that shows promise for applying conceptual change theory to the field of science education is Erilymaz's (2002) protocol for conceptual change discussion. In this model, teachers use challenging problems to identify students' alternative conceptions. This identification, termed 'exposure' by Erilymaz, involves predicting the results of scientific experiments as well as attempts at solving problems. Exposure is followed by conducting the scientific experiments, or 'discrepant events', during which students observe outcomes that conflict with their predictions. Teachers use questioning to help students focus on this discrepancy, increasing the level of cognitive conflict and encouraging students to grapple with the inconsistencies observed. At a definable point during this discussion cognitive conflict reaches disequilibrium, whereby students reject their initial predictions and accommodate the new information, addressing their own alternative conceptions in the process.

Erilymaz's approach, while originally designed for science educators, shares some common features with the Launch-Explore-Summarise (LES) structure designed by

Lappan, Frey, Fitzgerald, Friel and Phillips (2006) for using challenging tasks within mathematics lessons. Both approaches begin with the posing of a challenging problem. The explore phase from the LES structure, in which students experiment with and discuss ways to solve the mathematical problem posed is also mirrored in Erilymaz's structure, with scientific experimentation used as discrepant events to provoke student exploration and cognitive conflict. Both approaches also draw on student discussion, with a plenary summarizing phase led by the teacher that draws together different student ideas and formalizes knowledge. Just as challenging tasks following the LES structure have been found to encourage students to think deeply about mathematics and "connect different aspects of mathematics together, to devise solution strategies for themselves and to explore more than one pathway to solutions" (Sullivan, Askew, Cheeseman, Clarke, Mornane, Roche & Walker, 2014, p.6), so Erilymaz argues that posing problems combined with discrepant events in science lessons can encourage students to connect different concepts, explore ideas and change their own conceptions.

Longfield (2009), extends Erilymaz's work on discrepant events to mathematics and history lessons, concluding that these were useful for motivating students to re-examine their thinking, becoming active participants in their own learning and creating new knowledge for themselves. Longfield also suggests that this approach to conceptual change theory may be useful for addressing alternative conceptions in mathematics.

Theoretical approach

Within conceptual change theory three steps for learning are considered essential (for example see Mayer 2008, Luciarello 2014). The following three steps for learning form the basis for the lesson analysis in this report:

1. The learner recognizes an anomaly in their thinking, with a rising awareness that his/her current conception is inadequate to explain observable facts. This step involves using discrepant events to create cognitive conflict, thereby "motivating students to reexamine their thinking about previously held ideas and beliefs" (Longfield, 2009, p.266).
2. The learner actively constructs a new model that is able to explain the observable facts.
3. The learner uses the new model to find a solution to a problem.

Methodology

Eight mathematics lessons in which the researcher used the LES structure to pose and explore challenging problems with students in each of the grades from prep to grade seven were recorded using three video cameras and four microphones. As literature on how to assess conceptual change is limited (Jonassen, 2006), 20 incidents were selected from these lessons in which all of the following steps took place: an alternative conception was identified, discrepant events were juxtaposed with questioning to prompt accommodation, the student actively created a new model of understanding and the student generalized this model to solve both the initial challenging problem and a new, more difficult problem.

The selected incidents included:

Prep:	Three incidents: Two involving number conservation (to five) and one involving partitioning of single digit numbers.
Grade one:	One incident: Relative size of numbers to 10.
Grade two:	Three incidents: One involving number conservation and two involving partitioning of two-digit numbers into tens and ones.
Grade three:	Three incidents: One involving relative size to 10, one with relative size to 100 and one with relative size to 1000.
Grade 4:	Three incidents: One involving the equivalence of halves, one involving the comparative size of triangles and rectangles and one involving the use of the term "quarters" to mean any fraction that was not halves.
Grade 5:	Three incidents all involving decimal numbers: One in which students thought tenths were the same size as ones, one in which students thought tenths were the same size as halves, and one in which students thought that 0.7 was the same as 1/7.
Grade 6:	One incident: Involving the commutativity of multiplication.
Grade 7:	Three incidents all involving Proportional Reasoning: One involving the equivalence of halves and two involving the equivalence of thirds.

Table 1. Incidents Selected for Examination Using Cognitive Change Theory

For the purpose of this paper, discussion is limited to three incidents – one from each of Prep, Grade 3 and Grade 4. For each of the three incidents selected, the video and audio recordings were synced to create an audio-visual record that captured multiple viewpoints. Each incident was then transcribed, with additional annotations included for student actions as well as still images captured from the video feed to illustrate expressions and movements. Similarities between the incidents were noted and examined, to identify actions, questions, statements and expressions that met any of the criteria described in the previous paragraph. An analysis of the juxtaposition of discrepant events with questioning is included in the results section.

Results and Discussion

Data from the audio-visual record and transcripts indicates that the following five phases were present in each of the incidents analysed:

1. Phase 1: An alternative conception held by one or more students was identified by the researcher and confirmed at least twice using a challenging problem.
2. Phase 2: Students predicted the results of a discrepant event before it was carried out and then observed a different outcome to that predicted. The researcher juxtaposed multiple discrepant events (minimum 13 events) with questioning to build cognitive conflict.
3. Phase 3: The student discarded his or her alternative conception and accommodated the new information to create a new conceptual model. The researcher prompted students to explain this change in ideas.
4. Phase 4: The student used his or her new model to successfully solve the initial challenging problem.
5. Phase 5: The student generalized this new model to answer at least one more difficult question that required the new understanding.

Further analysis focused on the interplay between discrepant events and questioning and the creation of cognitive conflict during phases two and three of the selected incidents. Findings are summarised in Table Two below.

<table>
<tr><th>Discrepant Events Observed:</th><th>Most Commonly Occurring Questions:</th></tr>
<tr><td colspan="2">Incident 1: Prep students thought that five counters would no longer be five when they were moved. One student predicted four different amounts when the same five counters were shaken in a cup (initially six, then three, four and finally two). Following the incident one student stated, “It will still be the same amount – none fell out.” A second student added further explanation stating, “You’re not magic!”</td></tr>
<tr><td>13 discrepant events:
Shaking the counters in a cup, predicting how many there would be, tipping these out and counting them (5 times).
Placing five blocks on a desk and moving these around into different spatial arrangements. Predicting how many there would be, counting these and discussing why there were still five (4 times involving different arrangements, with multiple times counting each, 8 events altogether).</td><td>How many will there be now?
So you think the number of counters will change each time?
How many are there really – you count them for me?
Did it change?
It’s still five? But I thought it was going to change?
Did I shake it wrong? Is there something else I could try?
How come it didn’t change?
Is there a way that you could move the blocks so that there wouldn’t be five?</td></tr>
<tr><td colspan="2">Incident 2: Grade three students thought that 100 would be half way on a number line between one and 1000. The line was constructed from masking tape on the floor (5m long). Following the incident, the students successfully self-corrected their initial prediction and successfully placed several three-digit numbers on the line in their correct positions.</td></tr>
<tr><td>39 discrepant events:
Students were given 200, 300 and 400 to place on their line, then 900, 800, 700, 600 and 500. They observed that there was not enough space and moved the 100 to the ¼ position between one and 1000 (8 events).
Students and teacher stepped out the line, counting in hundreds to observe that there was a large space between the 1 and the 100. Students moved the 100 a little closer to the one, but maintained a relatively larger space between the one and 100 compared to the other hundreds (4 events).
Students were asked to explain why they left a larger space between one and 100. They stated that they were leaving room for the tens. Students were given 10, 20... 90 to place on their line. They moved the 100 back to the middle of the line (9 events).
Students were given 110, 120, 130 and 140 to place on their line. They expressed confusion and tried to move the 200 closer to the 1000. Students were given 210 and 310 to place on their line. Students stated that the line was too short (6 events).
The students counted in tens between each hundred, identifying how many tens were in each and considered if there was enough space for all of the tens (9 events).
Students removed the tens from the line and stepped it out again (3 events) before finally moving the 100 to the correct position.</td><td>Where do you think (this number) goes? Note: This question was asked more than 30 times with different numbers and in slightly different ways throughout the incident.
How about 200, 300, 400...900?
Does that look right to you? (Asked when students started to look quizzically at the line)
Which bit looks funny? How come it looks funny?
Can you make it look right please? You move the blocks until it looks right to you.
Tell me about this space?
How many tens are there in here? How many in here?
How about this space? Aren’t there any tens in here?
What do you notice about all the spaces?
How big is 100 compared to 1000? Is it a really big number? Is it about half way? Well where do you think it goes then?
That looks pretty different to where you originally had the 100 – tell me about why you changed your mind.</td></tr>
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<table>
<tr><td colspan="2">Incident 3: One grade four student thought that when two congruent, right-angled triangles were joined to make an isosceles triangle, this was larger than when the same two triangles were joined to form a rectangle. This focused on informal representations of area rather than calculating area of triangles. Following the incident the student solved the initial problem and also applied this solution to area problems for other shapes.</td></tr>
<tr><td>32 discrepant events

Both the isosceles triangle and rectangle were formed on the desk using four congruent, right-angled triangles (two for each shape). The boy thought the triangle was bigger. The pieces of the rectangle were shifted to form a second isosceles triangle identical to the first. The boy stated, “Now they’re the same”. This position was not maintained when the pieces were shifted back to form the two different shapes. The child explained this by stating, “The triangle is always bigger than the rectangle.” (2 events)
Swapping one of the right-angled triangular pieces from the rectangle with an identical piece from the triangle. The boy thought that the triangle would still be bigger (1 event).
The pieces of the rectangle were rearranged to form a parallelogram. The boy decided that this was confusing and picked up the triangle to cover the parallelogram so that he could compare them. He stated with some surprise, “They’re kind of the same size.” The teacher then rearranged the pieces in the triangle to exactly overlay the parallelogram, at which point the boy said, “Now they’re the same” (2 events).
The pieces were moved apart by 2cm. The boy decided that now they were smaller. The teacher drew his attention to look at the whole area by reminding him that he got to “Eat both bits of cake” in each situation. He maintained his position, saying, “They’d still be the same size if they’re together, but if they’re separated that means this one (touches the joined shape), that means this one’s bigger.” (2 events)
The teacher moved the separated pieces by tiny amounts, asking the boy to choose which was bigger each time until the pieces touched, forming the original triangle instead of a parallelogram. He maintained that the pieces were smaller until they touched, at which point this changed to “bigger”. The pieces were separated (“smaller”), then joined (“bigger”) multiple times (14 events).
Each piece was picked up and rotated rapidly in the air. The boy stated, “You’re just shifting it”. Both pieces were picked up and rotated simultaneously. He maintained that they were just being shifted. Then the pieces were joined together rapidly to form each of the different shapes examined so far. The boy stated, “You’re making it bigger”. The teacher drew his attention to this discrepancy by asking, “Bigger than the two pieces on their own?” While he answered, “Yes”, the pitch of his voice rose indicating that he was questioning his idea (6 events).
The paper was returned to the desk and shifted to form a variety of shapes. The teacher asked, “Am I changing the amount of paper?” The boy decided that the amount of paper wasn’t changing. Following some more pointed questions, the boy stated, “They’re the same size.” When the situation was changed he maintained, “This one looks bigger, but they’re the same size” (5 events).
</td>
<td>
Which one’s the biggest now?
How about if I shift them like this, which one’s the biggest now?
Now they’re the same? Ok, how about if I shift this one? Which is the biggest now?
This one’s bigger now? How about if I swap these bits?
The triangle’s bigger than this one?
How about if I move them like this?
So now they’re kind of the same?
What if I move the pieces apart?
So if the pieces are touching it’s the same size, but if they’re separated you think it’s smaller?
Now it’s bigger?
Now it’s smaller?
(picks up a piece and rotates it in the air) Am I changing the size of it?
How about this one? Am I changing the size of it?
How about if I put the pieces together, but you get to have both pieces either way?
So now it’s bigger than the two pieces on their own?
Am I changing the amount of paper?
So I have more paper if I move them like this?
How much paper is there?
Does it matter how I arrange the pieces? Does that change the amount of paper?
So you think they look different, but actually they’re the same size?
</td></tr>
</table>

Table 2. Discrepant Events, Questioning, Cognitive Conflict and Accommodation in Three Incidents During Which Students Discarded Their Alternative Conceptions

Of particular note to the researcher was the nature of the questioning within phases two and three. As a student's alternative conception was identified, the questions became more pointed, exposing the disparity between prediction and observation. During a 30-second discussion in the grade four incident, the teacher asked, "Now they're the same? Ok, how about if I shift this one? Which is the biggest now? This one's bigger now? How about if I swap these bits? The triangle's bigger than this one?" These questions required the student to make a choice and then observe the outcome of that choice. Successive questions formed sequences which appeared to narrow the available options and produce a logical process by which a student's idea could be evaluated. The questions in these phases both created and increased the cognitive conflict as a student confronted his or her own conceptions. At an identifiable moment during each incident this conflict appeared to peak, reaching a tipping-point of disequilibrium whereby the student acknowledged the disparity between his or her observations and preconceptions and then resolved this disparity by reconceptualising his or her own ideas.

Initially this use of narrow, pointed questions within an otherwise challenging lesson structure seemed incongruous. However, these questions appeared to provide students with a logical process for confronting and changing their own conceptions. If conceptual understanding links both surface and deep learning as learners construct their own understandings (Bereiter, 2002), perhaps approaching alternative conceptions from a connectionist perspective (Askew et. al, 1997) and integrating pointed questions and disparate events into a challenging problem provides a way forward.

Conclusion:

This paper presents data that indicates that conceptual change theory can be applied to challenging mathematics lessons, enabling students to change their own minds and alter their own mathematical conceptions. Discrepant events and questioning can be juxtaposed to create cognitive conflict, leading students to discard their alternative conceptions and accommodate new information. The five phases identified in this report were found to be common in twenty discussions across eight recorded lessons in which students who displayed alternative conceptions demonstrated cognitive change.

Some cautions are wise at this point, including considering who is doing the thinking – the teacher or the student. It is important within this process to ensure that the student is genuinely changing his or her own thinking, rather than leading student to the "right" answer. Balance and sensitivity is needed on the part of the teacher to ensure that questions scaffold student thinking only as much as is necessary to build cognitive conflict to the point of disequilibrium and achieve cognitive change. For alternative conceptions to be genuinely addressed, students need to change their own minds.

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